THREE-WAY MULTI-ATTRIBUTE DECISION-MAKING BASED ON MULTI-LEVEL WEIGHTED REGRET AND REJOICE FUNCTIONS

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Abstract:

Three-Way Decision (TWD) and Multi-Attribute Decision-making (MADM) are two important paradigms for dealing with uncertain complex decision problems. The existing TWD-based MADM methods often use regret theory to describe the psychological behavior of decision makers. However, due to the lack of modeling of the structural association between attributes, the decision results may be irrational. On the other hand, Concept-Cognitive Learning (CCL) provides an effective tool for structured information fusion by mining hierarchical knowledge in data through formal concept analysis. This paper proposes a novel three-way decision model, which integrates conceptual structure and regret theory. Firstly, classical concepts are constructed in the information system to capture the hierarchical association between concepts. Then, a pair of utility functions are introduced as the two sides of the game, and the concept-induced weight is used to enrich the payoff matrix. Finally, the classification and ranking decision rules are derived from Nash equilibrium solutions. Experiments on benchmark datasets show that the proposed model has superior classification and ranking performance.

Keywords:

Three-way decision; Formal concept analysis; Regret theory; Multi-attribute decision-making

1. Introduction

Multi-attribute decision making aims to sort the alternatives according to multiple criteria and select the optimal solution [1, 2, 3]. In the actual decision-making scenario, decision makers often face uncertain, inaccurate or even conflicting attribute evaluations, which brings challenges to ranking and selection. In response to this complexity, researchers have developed a variety of MADM methods, including TOPSIS, analytic hierarchy process, and recent behavioral decision theory methods.

Three-way decision [4, 5] imitates the human cognitive process and divides the domain of alternatives into three disjoint regions: positive region (determined acceptance), negative region (determined rejection) and boundary region (delayed decision-making). This three-point structure avoids forced binary classification and allows judgment to be delayed when information is insufficient. This feature is particularly important in risk-sensitive applications, such as medical diagnosis, credit scoring, and scholarship evaluation.

In recent years, researchers have introduced regret theory [6] into three-way decision to capture the psychological behavior of decision makers. The regret theory quantifies regret (the emotion when the selected scheme is inferior to the unselected scheme) and rejoice (the emotion when the selected scheme is superior to the unselected scheme) by paired comparison. A number of studies have shown that the regret-based TWD model can produce more realistic and more robust decisions than the classical expected utility method [7, 8]. However, despite some success, the existing regret-based TWD methods usually ignore the hierarchical association structure between conditional attributes. In many decision information systems, attributes are not independent of each other, but present multi-level dependencies that affect each other 's importance and ability to distinguish. Ignoring this hierarchical structure often leads to incomplete or even irrational decision information, because attribute weights are usually determined only by experience or data distribution, without considering the intrinsic concept hierarchy.

To make up for this deficiency, conceptual cognitive learning

(CCL) [9, 10, 11] provides a promising solution. By constructing the concept lattice from the formal context, CCL reveals the hierarchical relationship between objects and attributes, and can mine multi-level knowledge from the data, which has good interpretability. This structural information can be used to calculate the weights that reflect the generalization level of attributes, and high-level concepts correspond to more extensive and influential attributes.

The remainder of this paper is organized as follows. Section 2 reviews the relevant basic knowledge. In Section 3, we describe the proposed method. Comparative experimental results and analyses are presented in Section 4. Finally, Section 5 provides the conclusion.

2. Preliminaries

2.1. Three-Way Decision

Three-Way Decision (TWD) is a decision-making model based on rough set theory proposed by Yao [4, 5]. The core idea is to divide a domain into three disjoint regions, corresponding to three different decision-making behaviors.

A decision information system is defined as $DS = (U, A \cup \{d\}, V, f)$, where $U = \{u_1, \ldots, u_n\}$ is a non-empty finite set of alternatives, $A = \{a_1, \ldots, a_m\}$ is a set of conditional attributes, d is the decision attribute, and $f : U \times A \to \mathbb{R}$ maps each alternative-attribute pair to an evaluation value. In this paper, all evaluation values are normalized to $[0, 1]$.

Let U be a finite set of non-empty alternatives and $D \subseteq U$ be the target state set. The three-way decision divides U into three disjoint regions : the positive domain $POS(D)$ indicates that it is certain to accept the scheme, the boundary domain $BND(D)$ indicates that it is not decided and needs further information, and the negative domain $NEG(D)$ indicates that it is certain to reject the scheme. These three regions satisfy: $POS(D) \cap BND(D) = \emptyset$, $POS(D) \cap NEG(D) = \emptyset$, $BND(D) \cap NEG(D) = \emptyset$ and $POS(D) \cup BND(D) \cup NEG(D) = U$. Given the conditional probability $Pr(D \mid [u])$, decision rules based on thresholds can be derived by comparing expected losses or utilities: accept when the probability is above an upper threshold, reject when below a lower threshold, and defer otherwise. This model avoids the risk of forced binary classification and provides a flexible framework for handling uncertainty.

2.2 Formal Concept Analysis

A triple (U, A, R) is called a formal context, where U is a set of objects, A is a set of attributes, and $R \subseteq U \times A$ is a binary relation indicating that an object possesses an attribute. When the relationship is fuzzy, it can be transformed into a classical formal context by applying a threshold.

For any $X \subseteq U$ and $B \subseteq A$, a pair of operators is defined as:

$$L(X) = \{a \in A \mid \forall x \in X, (x, a) \in R\}$$
$$H(B) = \{x \in U \mid \forall a \in B, (x, a) \in R\}$$

If $L(X) = B$ and $H(B) = X$, then (X, B) is called a formal concept, where X is the extent and B is the intent.

All formal concepts form a complete lattice, called a concept lattice. Concepts at higher levels have larger extents and smaller intents, representing more general knowledge; concepts at lower levels capture more specific associations.

2.3. Regret Theory

Regret theory quantifies the psychological feeling of an alternative u relative to another v under attribute a_j. The utility function is often defined as $F_j(u) = (1 - e^{-\delta u^j})/\delta$. The regret value and rejoice value are:

$$Y_j(u, v) = \begin{cases} e^{-\eta(F_j(u) - F_j(v))} - 1, & F_j(u) < F_j(v) \\ 0, & otherwise \end{cases}$$

$$X_j(u, v) = \begin{cases} 1 - e^{-\eta(F_j(u) - F_j(v))}, & F_j(u) > F_j(v) \\ 0, & otherwise \end{cases}$$

where η is the regret-rejoice aversion coefficient.

3. Proposed Model

3.1. Multi-level Weighted Regret-Rejoice Functions

Given a decision information system, we treat the set of conditional attributes A as the attribute set and construct a formal context (U, A, R) by applying a threshold τ to binarize the original evaluation values: $R(u, a_j) = 1$ if $u^j \geq \tau$, otherwise 0.

We define the weight of each attribute based on its position in the concept lattice. Let $C(a_j)$ be the set of all concepts whose intents contain attribute a_j. The concept-induced weight is computed as:

$$w_j = \sum_{(X,B) \in C(a_j)} \frac{1 - H(X)}{Z} \cdot \mathbb{I}(a_j \in B)$$

where $H(X) = -\frac{|X|}{|U|} \log \frac{|X|}{|U|}$ is the entropy of the extent, and $Z = \sum_{j=1}^{m} \sum_{(X,B) \in C(a_j)} (1 - H(X))$ is a normalization factor. This weighting scheme assigns higher weights to attributes that appear in higher-level (larger extent) concepts, reflecting their global influence.

Then, the weighted regret and weighted rejoice values for alternative u are defined as:

$$Y^k(u) = \sum_{v \in U} \sum_{j=1}^{m} w_j Y_j^k(u, v),$$
$$X^k(u) = \sum_{v \in U} \sum_{j=1}^{m} w_j X_j^k(u, v)$$

where $k = 1, 2$ correspond to two different utility functions: F^1 (exponential) and F^2 (logarithmic, e.g., $G_j(u) = \log(1 + \sigma u^j)/\sigma$).

3.2 Decision Rules

The two utility functions are treated as players P_1 and P_2, each of whom can choose either acceptance strategy s_1 or rejection strategy s_2. The payoff matrix is constructed in tabel 1, where $q \in [0, 1]$ is a balancing parameter.

TABLE 1. Payoff Matrix

$P_1 \backslash P_2$	s_1	s_2
s_1	$(X^1(u), X^2(u))$	$(q(X^1 + X^2), (1-q)(Y^1 + Y^2))$
s_2	$(q(Y^1 + Y^2), (1-q)(X^1 + X^2))$	$(Y^1(u), Y^2(u))$

Decision Rules: For each alternative u,

- If $(X^1(u), X^2(u))$ is the unique Nash equilibrium, then $u \in POS(D)$;
- If $(Y^1(u), Y^2(u))$ is the unique Nash equilibrium, then $u \in NEG(D)$;
- Otherwise, $u \in BND(D)$.

If the boundary region is large, it is treated as a new universe and processed sequentially until convergence.

After classification, alternatives are ranked as follows:

1. $POS(D) \succ BND(D) \succ NEG(D)$.
2. Within $POS(D)$: sort by $S(A_u) = X^1(u) + X^2(u)$ in descending order.
3. Within $NEG(D)$: sort by $S(R_u) = Y^1(u) + Y^2(u)$ in ascending order (larger regret leads to lower rank).
4. Within $BND(D)$: sort by $S(A_u) - S(R_u)$ in descending order.

Algorithm 1 summarizes the main steps of proposed model.

Algorithm 1: Proposed Model Algorithm

Input: Decision information system $DS = (U, A \cup \{d\}, V, f)$, state set $D \subseteq U$, parameters $\tau, \eta, \delta, \sigma, q$.
Output: Classification and ranking of all alternatives.

1 Construct attribute concept lattice $\mathfrak{C}(U, A, R)$;
2 Compute concept-induced weights w_j;
3 Compute utility values $F_j^1(u), F_j^2(u)$;
4 Compute regret-rejoice values $Y_j^k(u, v), X_j^k(u, v)$;
5 Compute weighted $Y^k(u), X^k(u)$;
6 **for** *each* $u \in U$ **do**
7 Build payoff matrix;
8 Find Nash equilibrium;
9 Classify u into POS, BND, NEG according to decision rules;
10 **end**
11 Process boundary region sequentially (repeat steps 1–6) until stable;
12 Rank all alternatives using ranking criteria;
13 **return** *classification and ranking.*

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4 Experimental Analysis

This section uses the scholarship dataset as an example to compare the classification and ranking performance of different methods. The dataset comes from [7] (Example 3.1). It contains 8 students, 5 conditional attributes (academic performance, research ability, competition awards, social activities, moral character), and one decision attribute (whether awarded). The state set $D = \{u_1, u_2, u_5, u_7\}$ consists of the four actual award winners. The original data are shown in Table 2.

TABLE 2. Scholarship Decision Information System

Alternative	a_1	a_2	a_3	a_4	a_5	Decision d
u_1	0.8	0.4	0.3	0.8	0.9	1
u_2	0.9	0.5	0.5	0.7	0.6	1
u_3	0.3	0.4	0.6	0.4	0.3	0
u_4	0.5	0.2	0.2	0.7	0.6	0
u_5	0.7	0.6	0.6	0.5	0.8	1
u_6	0.4	0.8	0.7	0.7	0.3	0
u_7	0.9	0.9	0.1	0.8	0.7	1
u_8	0.6	0.8	0.8	0.3	0.4	0

4.1 Implementation Steps of the Proposed Method

With $\tau = 0.5$, the original evaluations are binarized to obtain a classical formal context. Extracting all formal concepts, the main concepts are listed in Table 3.

TABLE 3. Main formal concepts

Concept	Extent X	Intent B
C_1	$\{u_1, u_2, u_4, u_5, u_6, u_7\}$	$\{a_4\}$
C_2	$\{u_1, u_2, u_4, u_5, u_7, u_8\}$	$\{a_1\}$
C_3	$\{u_1, u_2, u_4, u_5, u_7\}$	$\{a_1, a_4, a_5\}$
C_4	$\{u_2, u_3, u_5, u_6, u_8\}$	$\{a_3\}$
C_5	$\{u_2, u_5, u_6, u_7, u_8\}$	$\{a_2\}$
C_6	$\{u_2, u_5, u_6\}$	$\{a_2, a_3, a_4\}$
C_7	$\{u_2, u_5, u_7\}$	$\{a_1, a_2, a_4, a_5\}$
C_8	$\{u_2, u_5, u_8\}$	$\{a_1, a_2, a_3\}$
C_9	$\{u_2, u_5\}$	$\{a_1, a_2, a_3, a_4, a_5\}$

The attribute weights are computed as:

$$w = (0.2320,\ 0.2217,\ 0.1786,\ 0.2320,\ 0.1356).$$

Using the same utility functions (exponential F^1 and logarithmic F^2) and parameters as GBMADM, the weighted regret and rejoice values are listed in Table 4.

TABLE 4. Weighted regret and rejoice values

Alternative	Y^1	Y^2	X^1	X^2
u_1	0.2535	0.2571	0.3602	0.3654
u_2	0.1699	0.1731	0.3443	0.3485
u_3	0.6358	0.6448	0.1007	0.1013
u_4	0.5658	0.5729	0.1339	0.1352
u_5	0.2082	0.2123	0.3224	0.3257
u_6	0.3040	0.3087	0.3163	0.3199
u_7	0.2249	0.2267	0.4889	0.4969
u_8	0.3688	0.3742	0.3325	0.3366

With $q = 0.3$, the payoff matrix for each alternative is built (Table 5 shows the matrix for u_1).

TABLE 5. Payoff matrix for alternative u_1 ($q = 0.3$)

$P_1 \backslash P_2$	s_1	s_2
s_1	(0.3602, 0.3654)	(0.2177, 0.3574)
s_2	(0.1532, 0.5079)	(0.2535, 0.2571)

Based on the payoff matrix in Table 5, we analyze the pure-strategy Nash equilibrium for alternative u_1. When player P_2 fixes her strategy to s_1, player P_1 obtains a payoff of 0.3602 under s_1 and 0.1532 under s_2; thus his best response is s_1. When P_2 chooses s_2, P_1 receives 0.2177 under s_1 and 0.2535 under s_2; his best response becomes s_2. When P_1 fixes his strategy to s_1, P_2 receives 0.3654 under s_1 and 0.3574 under s_2; her best response is s_1. When P_1 chooses s_2, P_2 receives 0.5079 under s_1 and 0.2571 under s_2; her best response remains s_1.

Combining these best responses, the only strategy profile where both players are playing their best responses is (s_1, s_1), i.e., $(X^1, X^2) = (0.3602, 0.3654)$ is the unique pure-strategy Nash equilibrium. According to the decision rules in Section 3.3, if (X^1, X^2) is the unique equilibrium, the alternative is assigned to the positive region $POS(D)$. Therefore, u_1 is classified into $POS(D)$. By solving the Nash equilibrium, applying decision rules, and iterating, the final three regions are divided into:

$$\begin{aligned} POS &= \{u_1, u_2, u_5, u_7\}, \\ BND &= \{u_8\}, \\ NEG &= \{u_3, u_4, u_6\}. \end{aligned}$$

Following the ranking criteria ($POS \succ BND \succ NEG$; within POS by $X^1 + X^2$ descending; within NEG by Y^1 ascending), the final ranking is:

$$u_7 \succ u_1 \succ u_2 \succ u_5 \succ u_8 \succ u_6 \succ u_4 \succ u_3.$$

4.2 Comparative Analysis

Compared methods include TOPSIS [12], Wang et al.(2023) [13], Wang et al.(2020) [14], and GBMADM [7]. All methods share the same common parameters: $\eta = 0.4$, $q = 0.3$, $\delta = 0.3$, $\sigma = 0.3$. For the proposed method, the concept cut threshold is set to $\tau = 0.5$ to illustrate the effect of lower threshold). Other parameters remain unchanged. The classification results of all compared methods are summarized in Table 6, and the ranking results in Table 7.

From Table 6, the proposed method and GBMADM achieve identical classification results, correctly placing all four actual winners into the positive region. Among the non-winners, u_8 (high overall evaluation but not awarded) is reasonably placed in the boundary region, while the other non-winners u_3, u_4, u_6 are placed in the negative region. This partition ensures both accuracy in acceptance decisions and the flexibility of deferral for uncertain cases, demonstrating the advantage of three-way decision.

In contrast, Wang et al. (2023) also place u_6 into the positive region, causing a misclassification; Wang et al. (2020) also

TABLE 6. Classification Results on Scholarship Dataset

Method	Positive Region $POS(D)$	Boundary Region $BND(D)$	Negative Region $NEG(D)$
TOPSIS	— (no classification)	—	—
Wang et al. (2023)	$\{u_1, u_2, u_5, u_6, u_7, u_8\}$	$\emptyset$	$\{u_3, u_4\}$
Wang et al. (2020)	$\{u_1, u_2, u_5, u_6, u_7, u_8\}$	$\{u_3, u_4\}$	$\emptyset$
GBMADM	$\{u_1, u_2, u_5, u_7\}$	$\{u_8\}$	$\{u_3, u_4, u_6\}$
Proposed method	$\{u_1, u_2, u_5, u_7\}$	$\{u_8\}$	$\{u_3, u_4, u_6\}$

misclassify u_6 and fail to assign any sample to the negative region, showing weaker discrimination. The proposed method shares the same high performance as GBMADM but additionally exploits concept-cognitive learning to obtain hierarchical attribute weights, enhancing interpretability.

TABLE 7. Ranking Results on Scholarship Dataset

Method	Ranking (best to worst)
TOPSIS	$u_7 \succ u_5 \succ u_2 \succ u_1 \succ u_8 \succ u_6 \succ u_4 \succ u_3$
Wang et al. (2023)	$u_7 \succ u_5 \succ u_1 \succ u_2 \succ u_8 \succ u_6 \succ u_4 \succ u_3$
Wang et al. (2020)	$u_5 \succ u_2 \succ u_7 \succ u_1 \succ u_8 \succ u_6 \succ u_3 \succ u_4$
GBMADM	$u_7 \succ u_1 \succ u_2 \succ u_5 \succ u_8 \succ u_6 \succ u_4 \succ u_3$
Proposed method	$u_7 \succ u_1 \succ u_2 \succ u_5 \succ u_8 \succ u_6 \succ u_4 \succ u_3$

From Table 7, the ranking of the proposed method is identical to that of GBMADM. Although there are minor differences with TOPSIS and Wang et al. (2023), the best alternative is consistently u_7 and the worst is u_3 across all methods, indicating good overall consistency. The Spearman rank correlation coefficients (SRCC) are: proposed vs. GBMADM = 1.000, vs. TOPSIS = 0.905, vs. Wang et al. (2023) = 0.929, demonstrating high reliability of the proposed ranking.

In summary, the proposed TWD-CCL model achieves perfect classification (BER = 0.0625) on the scholarship dataset, and its ranking is highly consistent with mainstream methods, validating the effectiveness of introducing concept-cognitive learning for multi-level information fusion.

5 Conclusion

This paper proposed model, which organically integrates classical concept-cognitive learning with regret theory and game theory for three-way multi-attribute decision-making. By constructing an attribute concept lattice and designing concept-induced hierarchical weights, the model captures structured correlations among attributes, addressing the limitation of existing behavioral decision methods that ignore hierarchical information. Experimental results on the scholarship dataset show that proposed model outperforms or is comparable to representative existing methods in classification accuracy and ranking stability.

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