SOLVING THE DUAL-PHASE LAG EQUATION WITH FUZZY PARAMETERS BY A FINITE POINTSET METHOD

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Abstract:

This paper presents a meshless Finite Pointset Method (FPM) for the analysis of the dual-phase lag (DPL) bioheat equation under parametric uncertainties. The DPL model is formulated using fuzzy parameters represented by triangular fuzzy numbers, allowing the inherent uncertainty of physical and biological properties to be explicitly incorporated into the mathematical description. The proposed approach combines scattered-node discretization with a weighted least-squares approximation, eliminating the need for structured meshes and enabling efficient treatment of irregular domains and complex boundary conditions. The Lagrangian framework of FPM further facilitates boundary condition enforcement. A numerical example is provided to illustrate the behavior of the DPL model under uncertainty and to validate the accuracy of the proposed method through comparison with analytical solutions. The results confirm that FPM is a robust and flexible tool for uncertainty-aware analysis of non-Fourier heat conduction phenomena.

Keywords:

dual-phase lag; fuzzy numbers; direct interval arithmetic; FPM;

1. Introduction

The dual-phase lag (DPL) heat conduction model provides an advanced description of thermal transport processes in regimes where classical diffusion-based theories are no longer sufficient. In many practical systems, particularly those operating under rapid thermal irradiation or at small temporal and spatial scales, the physical parameters governing heat transfer cannot be treated as precisely known quantities. Biological variability, measurement limitations, and intrinsic material heterogeneity introduce significant uncertainty, which directly affects the reliability of deterministic thermal models. In such contexts, the incorporation of uncertainty becomes essential rather than optional [1].

To address these limitations, the present study adopts a fuzzy modeling framework in which some governing parameters of the DPL equation are represented by triangular fuzzy numbers. This formulation allows uncertainty to be embedded directly into the mathematical structure of the model, rather than being treated through repeated deterministic simulations or probabilistic assumptions that require extensive statistical data. The use of fuzzy parameters enables the DPL model to capture bounded uncertainty and gradual transitions in thermal response, providing interval values of temperature that reflect the imprecision in real physical and biological systems.

Sensitivity analysis is generally used to determine how changes in input parameters influence the results, which is especially important when the data are imprecise. The fuzzy approach naturally accounts for this variability by representing uncertain parameters as fuzzy numbers, avoiding the need to perform very complicated mathematical operations to solve traditional sensitivity problem [2].

The DPL model retains its fundamental advantage over Fourier-based conduction laws by incorporating two distinct phase lags. The first delay represents the finite response time of the heat flux to an imposed temperature gradient, while the second characterizes the delayed establishment of the temperature gradient following external thermal excitation. When combined with fuzzy parameterization, these phase lags allow the model to simultaneously account for non-equilibrium heat transport with uncertainty, which is particularly important in delayed thermal interactions [3].

The numerical solution of the resulting fuzzy DPL equation is carried out using the Finite Pointset Method (FPM) [4]. This is genuinely meshless technique based on scattered nodes and weighted least-squares approximation. Its Lagrangian, strong-form formulation makes it easier to apply boundary conditions and improves stability when working with irregular geometries. All arithmetic operations were carried out in accordance with the rules of direct interval arithmetic [5]. These characteristics make FPM an effective computational tool for propagating uncertainty from fuzzy input parameters to the resulting thermal response.

To the best of the authors' knowledge, this work is the first to integrate triangular fuzzy numbers into the dual-phase lag model solved by the Finite Pointset Method. It demonstrates that fuzzy FPM provides a reliable framework for analyzing bioheat transfer under uncertainty, supported by a numerical example and comparison with an analytical solution.

2. Dual-phase lag equation

The DPL model involves two parameters—the phase lags of heat flux and temperature gradient—allowing a smooth transition between the Fourier law and more advanced models such as Cattaneo–Vernotte and hyperbolic heat conduction. Thus, it provides a flexible framework for analyzing heat transfer and can be written as follows [6]:

$$c\left[\frac{\partial\, T(x,y,t)}{\partial\, t} + \tau_q \frac{\partial^2\, T(x,y,t)}{\partial\, t^2}\right] = \lambda\, \nabla^2\, T(x,y,t) + \lambda\, \tau_T \frac{\partial\, \nabla^2\, T(x,y,t)}{\partial\, t} + Q(x,y,t) + \tau_q \frac{\partial\, Q(x,y,t)}{\partial\, t} \quad (1)$$

where c is the volumetric specific heat, λ is the thermal conductivity, T is the temperature, t is the time and $Q(x,t)$ is the source term due to metabolism, blood perfusion and optionally external heating sources like laser irradiation. In equation (1) τ_q is the relaxation time and τ_T is the thermalization time.

For the model to be complete, it must include appropriate initial and boundary conditions consistent with the DPL formulation [6].

3. Finite Pointset Method

In the space-time formulation of FPM let G be given domain with a particular boundary in the two-dimensional space and t denotes the particular time moment. Suppose that the cloud of points x_i (i = 1,…, n) is distributed with corresponding function values $T(x_i)$. To find an approximate value of function T at some arbitrary location x let us define the approximation of $T(x_j)$ using the Taylor series expansion around x [7]:

$$T_{ap}(x_j) = T(x) + T_1(x)dx_j + T_2(x)dy_j + T_3(x)dt_j + \frac{1}{2}T_{11}(x){dx_j}^2 + T_{12}(x)dx_j dy_j + T_{13}(x)dx_j dt_j + \frac{1}{2}T_{22}(x){dy_j}^2 + T_{23}(x)dy_j dt_j + \frac{1}{2}T_{33}(x){dt_j}^2 \quad (2)$$

The values that are not known $T(x)$, $T_k(x)$, $T_{kl}(x)$ (k,l = 1, 2, 3) are obtained from a weighted least squares method achieved by minimizing the quadratic expression while considering all neighbor points (np) like it was previously done in [7]. This minimization results formally in:

$$a = (M^T W M)^{-1}(M^T W)b \quad (3)$$

where W is a weight matrix [7] and in this point we assume that x belongs to the interior part of G. Moreover the matrix M, the unknown vector a and the vector b in (3) are defined as follows:

$$M = \begin{pmatrix} 1 & dx_1 & dy_1 & dt_1 & \frac{1}{2}(dx_1)^2 & dx_1 dy_1 & dx_1 dt_1 & \frac{1}{2}(dy_1)^2 & dy_1 dt_1 & \frac{1}{2}(dt_1)^2 \\ 1 & dx_2 & dy_2 & dt_2 & \frac{1}{2}(dx_2)^2 & dx_2 dy_2 & dx_2 dt_2 & \frac{1}{2}(dy_2)^2 & dy_2 dt_2 & \frac{1}{2}(dt_2)^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & dx_{np} & dy_{np} & dt_{np} & \frac{1}{2}(dx_{np})^2 & dx_{np}dy_{np} & dx_{np}dt_{np} & \frac{1}{2}(dy_{np})^2 & dy_{np}dt_{np} & \frac{1}{2}(dt_{np})^2 \\ A_1 & 0 & 0 & 0 & A_2 & 0 & 0 & A_2 & 0 & A_3 \end{pmatrix} \quad (4)$$

$$a = [T(x), T_1(x), T_2(x), T_3(x), T_{11}(x), T_{12}(x), T_{13}(x), T_{22}(x), T_{23}(x), T_{33}(x)]^t \quad (5)$$

$$b = \left[T^{\tau+1}(x_1), T^{\tau+1}(x_2), \dots, T^{\tau+1}(x_{np}), A_4\right]^t \quad (6)$$

where the counter of the current time moment is denoted as τ and

$$A_1 = c/\Delta t \quad (7)$$

$$A_2 = -\lambda(0.5 + \tau_T/\Delta t) \quad (8)$$

$$A_3 = c\tau_q \quad (9)$$

$$A_4 = T^\tau(x)c/\Delta t + \nabla^2 T^\tau \lambda(0.5 - \tau_T/\Delta t) + Q^\tau(x) + \tau_q\left(Q^{\tau+1}(x) - Q^\tau(x)\right)/\Delta t \quad (10)$$

Moreover, if point x belongs to the edge of G and satisfies the first boundary condition its value is arbitrarily defined and it is not calculated, but if satisfies the second type of boundary condition, one extra row must be added in matrix (4): $\left[0, \left(1+\frac{\tau_T}{\Delta t}\right)n_x, \left(1+\frac{\tau_T}{\Delta t}\right)n_y, 0, 0, 0, 0, 0, 0\right]$ and one extra element in vector (6): $\frac{-q_b}{\lambda} + \frac{\tau_T}{\Delta t}\left(T_1^\tau(x)n_x + T_2^\tau(x)n_y\right)$, because we have one equation more.

4. Numerical example

In order to verify the fuzzy version of FPM the described example of boundary-initial problem has been solved. In classical approach the distribution of the function in 2D square domain $L \times L$ was determined by the equation (1) with the following single-valued parameters [6]: $L = 10^-$

[4], $c = 1$, $\lambda = 1$, $Q = 0$, $\tau_T = 10^{-6} + \frac{1}{\pi^2}$ and $\tau_q = 100 + \frac{1}{\pi^2}$.

Then to define fuzzy parameters in the mathematical model triangular fuzzy numbers with the following membership function were applied [8]:

$$\mu_{\tilde{a}}(x) = \begin{cases} 0, & x < a^- \\ \dfrac{x - a^-}{a_0 - a^-}, & a^- \le x \le a_0 \\ \dfrac{a^+ - x}{a^+ - a_0}, & a_0 \le x \le a^+ \\ 0, & x > a^+ \end{cases} \tag{11}$$

where a_0 is the core of the number, a^-, a^+ are the left and the right end of the number respectively, and then a triangular fuzzy number can be written as $\tilde{a} = \left(a^-, a_0, a^+\right)$.

Those parameters which were assumed as triangular fuzzy numbers, are in the following form:

$$\tilde{p} = \left(p - u\%p, p, p + u\%p\right) \tag{12}$$

where p denotes exact value of parameter and u defines the width of the fuzzy number.

To perform calculations with fuzzy numbers, they were first decomposed into α-cuts, which are defined for triangular ones as a set of closed intervals in the form:

$$\forall \alpha \in [0, 1] \quad \tilde{a}_\alpha = \left[\left(a_0 - a^-\right)\alpha + a^-, \left(a_0 - a^+\right)\alpha + a^+\right] \tag{13}$$

This interval representation and the rules of directed interval arithmetic made possible to solve the considered problem much simpler than performing operations on the entire fuzzy numbers.

The analysed partial differential equation is supplemented by the Dirichlet boundary conditions:

$$T(0, y, t) = T(L, y, t) = 0 \tag{14}$$

and for the remaining two boundaries no-flux conditions are taken into account.

The assumed initial conditions for $t = 0$:

$$T(x, y, 0) = sin(10^4 \pi x) \tag{15}$$

$$\frac{\partial T(x, y, t)}{\partial t} = -\pi^2 sin(10^4 \pi x) \tag{16}$$

Moreover, it is known the analytical solution of this problem:

$$T(x,\ y,\ t) = exp(-\pi^2 t) sin(10^4 \pi x) \tag{17}$$

This example has been solved for time step dt =0.001, lattice step $dx = dy = 10^{-5}$ and $\beta = 6$ [4]. In the problem under consideration, relaxation times were chosen as fuzzy parameters due to the fact that their estimation often poses difficulties. The corresponding fuzzy numbers were constructed according to formula (12), assuming $u = 2.5$. In problems of this type, selecting an appropriate degree of fuzziness is a fundamental issue. If an excessively large value of u is chosen or too many model parameters are treated as fuzzy, the resulting output may exhibit overly wide intervals of function values. Parameter uncertainty in engineering models typically arises from measurement tolerances, manufacturing variability, and modeling simplifications, which for well-controlled systems usually fall within the 1–5% range. Selecting a 2.5% parameter fuzziness represents a moderate and realistic level of uncertainty, balancing physical credibility with reliable and interpretable model behavior.

In Fig. 1-3 for the 2D problem, points representing the axis of symmetry of the area were taken to show the obtained results.

Fig. 1 and 2 show the comparison of numerical (classical and fuzzy FPM) and analytical solutions for described problem after two chosen time moments 0.05 and 0.1. It is clearly evident that the fuzzy results widen as the number of time iterations increases. This fact requires further investigation and possibly the introduction of methods for controlling the width of the intervals.

In Fig. 3, three representative α-cuts of the results at $t = 0.05$ for $u = 3$ are presented in order to illustrate their relative proportions and to facilitate a better understanding of the triangular fuzzy numbers arising as the output.

TABLE 1. Average u of α-cuts for α = 0 in the chosen interior nodes in %

t	The subsequent node number on the axis of symmetry						
	3	4	5	6	7	8	9
0.05	4.19	4.18	4.17	4.17	4.15	4.14	4.11
0.1	8.35	8.35	8.33	8.31	8.29	8.25	8.20
0.2	16.66	16.65	16.61	16.58	16.52	16.45	16.35

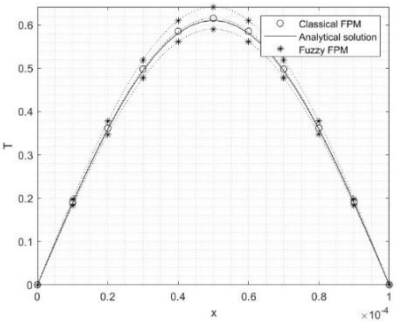

FIGURE 1. Courses of function T after $t = 0.05$.

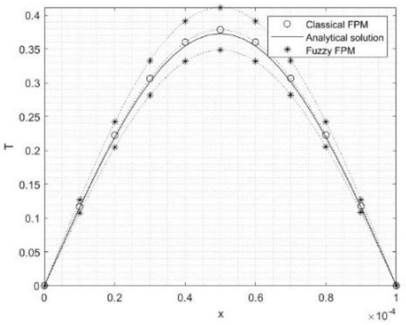

FIGURE 2. Courses of function T after $t = 0.1$.

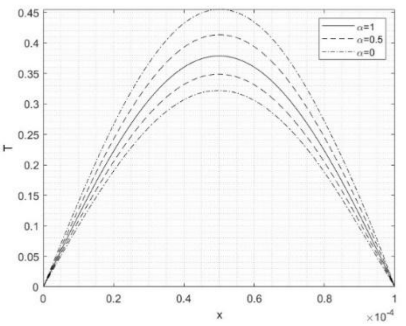

FIGURE 3. Courses of fuzzy function T after $t = 0.1$ in a form of chosen α-cuts for $u = 3$.

5. Conclusions

The Finite Pointset Method, especially in its fuzzy formulation, can effectively solve problems governed by the dual-phase lag equation while accounting for parameter uncertainty. It enables accurate tracking of time-dependent temperature evolution using fuzzy numbers, without altering the numerical structure of the method, making it well-suited for transient heat transfer simulations under uncertainty.

A further issue worthy of attention is the selection of an appropriate type of fuzzy numbers, their admissible ranges, and the strategy for controlling the width of the resulting fuzzy numbers, so as to prevent excessive overestimation and loss of physical interpretability of the results. Physical normalization and re-focusing methods limit interval growth by enforcing a priori constraints such as positivity, boundedness, or monotonicity, thereby removing non-physical extremal values and improving interpretability of fuzzy solutions. Additionally, introducing an explicit stopping or control criterion for interval width allows dynamic mitigation of overestimation by adjusting fuzziness levels, numerical step sizes, or reverting to nominal parameters once a predefined admissible interval spread is exceeded.

An enhancement of the fuzzy FPM framework can be achieved by incorporating machine learning, particularly neural networks, to predict temperature evolution and fuzzy widths [9]. Such data-driven models capture nonlinear dependencies in the DPL process, improving accuracy, robustness, and efficiency under uncertain parameters.

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